

# Weierstrass quasi-interpolants

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## Abstract

In this paper, the expression of Weierstrass operators as differential operators on polynomials is used for the construction of associated quasi-interpolants. Then the convergence properties of these operators are studied.

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**Keywords:** Approximation by integral operators.

## 1 Introduction

In the present paper, the general method developed by the author [19, 20, 21] for the construction of new quasi-interpolants from classical linear approximation operators is applied to Weierstrass operators. Similar studies are done in [14] for Bernstein operators and in [15] for Baskakov quasi-interpolants (abbr. QIs). The latter is completed in [22]. The idea can be summarized as follows : let  $\{Q_n, n \in \mathbb{N}\}$  be a sequence of linear operators defined on some functional space  $\mathcal{F}$  with values in a finite-dimensional subspace  $\mathcal{P}_n$  of algebraic or trigonometric polynomials. Assuming that for all  $n \in \mathbb{N}$ ,  $Q_n$  is an isomorphism of  $\mathcal{P}_n$  preserving the degree, i.e.  $Q_n p \in \mathcal{P}_m$  for any  $p \in \mathcal{P}_m, 0 \leq m \leq n$ , very often it admits a representation in that space as a linear differential (or a difference) operator of the form

$$Q_n = \sum_{r=0}^n \beta_r^{(n)}(x) \mathcal{D}_r$$

where  $\mathcal{D}_r$  is a linear differential (or difference) operator, and  $\beta_r^{(n)}(x)$  is a polynomial of degree at most  $r$ . In most cases, the inverse  $P_n := Q_n^{-1}$  of  $Q_n$  has also a representation of the same form

$$P_n = \sum_{r=0}^n \alpha_r^{(n)}(x) \mathcal{D}_r$$

In general, both families of polynomial coefficients satisfy a recurrence relation. This has been proved ([6, 7, 8, 9, 21]) for Bernstein and Szász-Mirakyan operators and their associated Durrmeyer versions, and also for Kantorovich operators.

Introducing the truncated inverses of order  $0 \leq r \leq n$ :

$$P_n^{(r)} = \sum_{k=0}^r \alpha_k^{(n)}(x) \mathcal{D}_k$$

one can associate with  $Q_n$  the family of (left) quasi-interpolants  $\{Q_n^{(r)}, 0 \leq r \leq n\}$  defined by

$$Q_n^{(r)} p := P_n^{(r)} Q_n p = \sum_{k=0}^r \alpha_k^{(n)}(x) \mathcal{D}_k Q_n p, \quad \forall p \in \mathcal{P}_n$$

By construction, this operator is exact on  $\mathcal{P}_r$  and it can be extended to the functional space  $\mathcal{F}$ . By virtue of classical theorems in approximation theory, this procedure greatly improves the convergence order of the initial operator  $Q_n$  in the space  $\mathcal{F}$ .

Since 1990, the method has been extended by several authors to various operators. For example by M.W. Müller [16] to Gamma operators, by A.T. Diallo [6, 8, 9] and others to Szász-Mirakyan operators. In this paper, the expression of Weierstrass operators as differential operators on polynomials is used for the construction of associated quasi-interpolants. Then the convergence properties of these operators are studied. These operators were first used by Weierstrass for the proof of the uniform approximation of continuous functions by polynomials.

Here is an outline of the paper. In Section 2, Weierstrass operators are expressed as differential or difference operators on polynomials. In Section 3, similar expressions are given for the inverse Weierstrass operators. In Section 4, Weierstrass left quasi-interpolants are defined on the space  $\Pi$  of polynomials. In the same way, in Section 5, Weierstrass right quasi-interpolants are defined on the same space. In Section 6, Weierstrass quasi-interpolants are expressed in terms of polynomials derived from Hermite polynomials. In Section 7 are studied the norms of the Weierstrass left quasi-interpolants and their convergence properties. Finally, in Section 8, some methods for the effective computation of these operators are briefly described. They will be detailed in a further paper.

## 2 Weierstrass operator as differential or difference operator on polynomials

Setting  $g_n(u) := \sqrt{\frac{n}{\pi}} \exp(-nu^2)$ , the Weierstrass operator ([3],[5] p.15) is defined as

$$W_n f(x) := (f * g_n)(x) = \int_{-\infty}^{+\infty} f(t) g_n(x-t) dt = \int_{-\infty}^{+\infty} f(x-t) g_n(t) dt$$

The first integral is called the *first form* of the Weierstrass operator

$$W_n f(x) = \sqrt{\frac{n}{\pi}} \int_{-\infty}^{+\infty} \exp(-n(x-t)^2) f(t) dt$$

Similarly, the second integral is called the *second form* of the Weierstrass operator. It can also be written, with the change of variable  $t = s/\sqrt{n}$ :

$$W_n f := \sqrt{\frac{n}{\pi}} \int_{-\infty}^{+\infty} f(x-t) \exp(-nt^2) dt = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} f(x-s/\sqrt{n}) \exp(-s^2) ds$$

It is linked to the heat kernel [12]  $K(t, x) := \frac{1}{(4\pi t)^{1/2}} e^{-x^2/4t}$  by the relation  $g_n(u) = K(\frac{1}{4n}, x)$ .

## 2.1 $W_n$ as differential operator

According to formulas (7.4.4) and (7.4.5) of [1], we have

**Lemma 1.** (i) *The even moments of the function  $\exp(-x^2)$  are given by*

$$M_{2k} := \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-s^2} s^{2k} ds = \frac{1}{\sqrt{\pi}} \Gamma(k + 1/2)$$

(ii) *The absolute odd moments of the function  $\exp(-x^2)$  are given by*

$$M_{2k+1} := \frac{2}{\sqrt{\pi}} \int_0^{+\infty} e^{-s^2} s^{2k+1} ds = \frac{1}{\sqrt{\pi}} \Gamma(k + 1)$$

Denoting monomials by  $m_p(x) = x^p$ , then we first have  $W_n m_0 = m_0$  and  $W_n m_1 = m_1$ . Then, we get successively:

$$W_n m_2 = m_2 + \frac{1}{2n} m_0, \quad W_n m_3 = m_3 + \frac{3}{2n} m_1, \quad W_n m_4 = m_4 + \frac{3}{n} m_2 + \frac{3}{4n^2} m_0,$$

$$W_n m_5 = m_5 + \frac{5}{n} m_3 + \frac{15}{4n^2} m_1, \quad W_n m_6 = m_6 + \frac{15}{2n} m_4 + \frac{45}{4n^2} m_2 + \frac{15}{8n^3} m_0$$

From that, we easily deduce the first terms of the expansion of  $W_n$  as differential operator on polynomials

$$W_n p = p + \frac{1}{4n} D^2 p + \frac{1}{32n^2} D^4 p + \frac{1}{384n^3} D^6 p + \dots$$

More generally, we get

**Theorem 1.** *The representation of  $W_n$  as differential operator on  $\Pi$  is the following:*

$$W_n = I + \sum_{k \geq 1} \frac{1}{4^k k!} \frac{1}{n^k} D^{2k}$$

*Proof.* By definition,

$$W_n m_r(x) = \frac{\sqrt{n}}{\sqrt{\pi}} \int_{\mathbb{R}} (x-t)^r e^{-nt^2} dt = \sum_{j=0}^r (-1)^j \binom{r}{j} x^{r-j} \frac{\sqrt{n}}{\sqrt{\pi}} \int_{\mathbb{R}} t^j e^{-nt^2} dt$$

and, using the change of variable  $s = \sqrt{n}t$ , we get

$$W_n m_r(x) = \sum_{j=0}^r (-1)^j \binom{r}{j} x^{r-j} n^{-j/2} \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} s^j e^{-s^2} ds$$

Finally, we obtain the desired result:

$$W_n m_r(x) = \sum_{k=0}^{\lfloor r/2 \rfloor} \binom{r}{2k} x^{r-2k} n^{-k} M_{2k} = \sum_{k=0}^{\lfloor r/2 \rfloor} \frac{1}{4^k k!} \frac{1}{n^k} D^{2k} m_r \quad \text{q.e.d. } \square$$

**Remark.** The formal power series (abbr. fps)  $\sum_{k \geq 0} X^{2k}/k!4^k n^k$  is the expansion of the function  $\exp(X^2/(4n))$ , therefore the differential form of the Weierstrass operator can be written

$$W_n = \exp(D^2/(4n)) = \sum_{k \geq 0} \frac{D^{2k}}{4^k k! n^k}$$

## 2.2 $W_n$ as difference operator

Let  $\delta f(x) = f(x+h/2) - f(x-h/2)$ . We start from the formula giving derivatives in terms of centered finite differences (see [2])

$$D^p f(x)/p! = \sum_{k \geq p} t(k, p) \delta^{2k} f(x)/(2k)!$$

where the coefficients  $t(k, p)$  are the *central factorial numbers* (abbr. cf<sub>n</sub>) of the first kind defined by

$$x^{[n]} := \sum_{k=0}^n t(n, k) x^k$$

with  $x^{[0]} = 1$ ,  $x^{[1]} = x$ , and  $x^{[n]} = x \prod_{k=1}^{n-1} (x + \frac{n}{2} - k)$  for all  $n \geq 2$ . Then we obtain

$$\begin{aligned} W_n &= \sum_{k \geq 0} \frac{D^{2k}}{(4n)^k k!} = \sum_{k \geq 0} \frac{(2k)!}{(4n)^k k!} \left( \sum_{\ell \geq k} t(2\ell, 2k) \frac{\delta^{2\ell}}{2\ell!} \right) \\ &= \sum_{\ell \geq 0} \left( \sum_{k \leq \ell} \frac{(2k)!}{(4n)^k k!} t(2\ell, 2k) \right) \frac{\delta^{2\ell}}{2\ell!} \end{aligned}$$

that we can write

$$W_n = Id + \sum_{\ell \geq 1} \beta_\ell(n) \frac{\delta^{2\ell}}{2\ell!}, \quad \text{with} \quad \beta_\ell(n) = \sum_{k=1}^{\ell} \frac{(2k)!}{(4n)^k} t(2\ell, 2k) \quad \text{for } \ell \geq 1$$

The first coefficients being

$$\begin{aligned}\beta_1 &= \frac{1}{2n}, & \beta_2 &= -\frac{1}{2n} + \frac{3}{4n^2}, & \beta_3 &= \frac{2}{n} - \frac{15}{4n^2} + \frac{15}{8n^3} \\ \beta_4 &= -\frac{18}{n} + \frac{147}{4n^2} - \frac{105}{4n^3} + \frac{105}{16n^4} \\ \beta_5 &= \frac{288}{n} - \frac{615}{n^2} + \frac{4095}{8n^3} - \frac{1575}{8n^4} + \frac{945}{32n^5}\end{aligned}$$

we get the first term of the expression of  $W_n$  as difference operator :

$$W_n = I + \frac{1}{2n} \frac{\delta^2}{2} + \left(-\frac{1}{2n} + \frac{3}{4n^2}\right) \frac{\delta^4}{4!} + \left(\frac{2}{n} - \frac{15}{4n^2} + \frac{15}{8n^3}\right) \frac{\delta^6}{6!} + \dots$$

### 3 The inverse $W_n$ -operator as differential or difference operator on polynomials

#### 3.1 Inverse of the Weierstrass operator as differential operator on $\Pi$

We look for the inverse of Weierstrass operator under the form

$$V_n := W_n^{-1} = I + \sum_{\ell \geq 1} d_\ell D^{2\ell}$$

The problem consists in computing the inverse  $V_n(X)$  of the fps  $W_n(X)$ :

$$W_n(X) = 1 + \sum_{k \geq 1} c_k X^k, \quad V_n(X) = 1 + \sum_{\ell \geq 1} d_\ell X^\ell, \quad V_n(X)W_n(X) = 1$$

As  $W_n = \exp(D^2/(4n))$ , it is straightforward to deduce  $V_n = \exp(-D^2/(4n))$ , thus

**Theorem 2.** *The inverse of  $W_n$  as differential operator on  $\mathbb{P}$  is given by*

$$V_n = Id + \sum_{k \geq 1} \frac{(-1)^k D^{2k}}{(4n)^k k!}$$

#### 3.2 Inverse of the Weierstrass operator as difference operator on $\Pi$

We use the formalism of Section 2.2:

$$\begin{aligned}V_n &= \sum_{k \geq 0} \frac{(-1)^k D^{2k}}{(4n)^k k!} = \sum_{k \geq 0} (-1)^k \frac{(2k)!}{(4n)^k k!} \left( \sum_{\ell \geq k} t(2\ell, 2k) \frac{\delta^{2\ell}}{2\ell!} \right) \\ &= \sum_{\ell \geq 0} \left( \sum_{k \leq \ell} (-1)^k \frac{(2k)!}{(4n)^k k!} t(2\ell, 2k) \right) \frac{\delta^{2\ell}}{2\ell!}\end{aligned}$$

that we can write

$$V_n = \sum_{\ell \geq 0} \alpha_\ell(n) \frac{\delta^{2\ell}}{2\ell!}, \quad \text{with} \quad \alpha_\ell(n) = \sum_{k=1}^{\ell} (-1)^k \frac{(2k)!}{(4n)^k k!} t(2\ell, 2k).$$

The first coefficients  $\alpha_\ell$  are

$$\begin{aligned} \alpha_1 &= -\frac{1}{2n}, \quad \alpha_2 = \frac{1}{2n} + \frac{3}{4n^2}, \quad \alpha_3 = -\left(\frac{2}{n} + \frac{15}{4n^2} + \frac{15}{8n^3}\right) \\ \alpha_4 &= \frac{18}{n} + \frac{147}{4n^2} + \frac{105}{4n^3} + \frac{105}{16n^4} \\ \alpha_5 &= -\left(\frac{288}{n} + \frac{615}{n^2} + \frac{4095}{8n^3} + \frac{1575}{8n^4} + \frac{945}{32n^5}\right) \end{aligned}$$

## 4 Left quasi-interpolants on polynomials

### 4.1 Weierstrass left-quasi-interpolants : first form

**Definition.** *Considering the partial sums of order  $r$  of the inverse  $W$ -operator*

$$V_n^{[r]} := \sum_{k=0}^r \frac{1}{n^k} \frac{(-1)^k}{4^k k!} D^{2k}$$

*one defines the Weierstrass left quasi-interpolants of order  $r$  as follows:*

$$W_n^{[r]} := V_n^{[r]} W_n := \sum_{k=0}^r \frac{1}{n^k} \frac{(-1)^k}{4^k k!} D^{2k} W_n, \quad 0 \leq r \leq n.$$

Using the first form of  $W_n f$ , we get

$$W_n^{[r]} f(x) = \sqrt{\frac{n}{\pi}} \int_{-\infty}^{+\infty} V_n^{[r]}[\exp(-n(x-t)^2)] f(t) dt$$

We thus need the expressions of derivatives of  $g_n(x-t)$  in terms of Hermite polynomials. From the definition of these polynomials (see e.g. [10, 13]), it is well known that

$$H_k(x) = (-1)^k e^{x^2} D^k e^{-x^2} \quad \Rightarrow \quad D^{2k} e^{-x^2} = e^{-x^2} H_{2k}(x)$$

Therefore, introducing the new polynomials

$$\tilde{H}_{2r}(s) := \sum_{k=0}^r \frac{(-1)^k}{4^k k!} H_{2k}(s),$$

we obtain

$$\begin{aligned} V_n^{[r]}[\exp(-n(x-t)^2)] &= \sum_{k=0}^r \frac{1}{n^k} \frac{(-1)^k}{4^k k!} D^{2k}[\exp(-n(x-t)^2)] \\ &= \sum_{k=0}^r \frac{(-1)^k}{4^k k!} H_{2k}(x-t) \exp(-n(x-t)^2) = \tilde{H}_{2k}(x-t) \exp(-n(x-t)^2) \end{aligned}$$

We then deduce

**Theorem 3.** *The quasi-interpolant  $W_n^{[r]}$  can be written under the two following forms*

$$\begin{aligned} W_n^{[r]}f(x) &= \sqrt{\frac{n}{\pi}} \int_{-\infty}^{+\infty} \tilde{H}_{2k}(x-t) \exp(-n(x-t)^2) f(t) dt \\ W_n^{[r]}f(x) &:= \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \exp(-s^2) \tilde{H}_{2r}(s) f(x - s/\sqrt{n}) ds \end{aligned}$$

The second expression is obtained by using the change of variable  $t = s/\sqrt{n}$ .

## 4.2 The polynomials $\tilde{H}_r$

Here are the expressions of the first polynomials  $\tilde{H}_{2r}$ , with  $\tilde{H}_0 = 1$ :

$$\begin{aligned} \tilde{H}_2(x) &= \frac{3}{2} - x^2, \quad \tilde{H}_4(x) = \frac{15}{8} - \frac{5}{2}x^2 + \frac{1}{2}x^4 \\ \tilde{H}_6(x) &= \frac{35}{16} - \frac{35}{8}x^2 + \frac{7}{4}x^4 - \frac{1}{6}x^6. \end{aligned}$$

More generally, one has

**Theorem 4.** *The general expression of polynomials  $\tilde{H}_{2r}$  is the following*

$$\tilde{H}_{2r}(x) = \frac{(2r+1)!}{r!} \sum_{p=0}^r \frac{(-1)^{r-p}}{4^p p!} \frac{x^{2(r-p)}}{(2r-2p+1)!}$$

*Proof.* From the definition:

$$\tilde{H}_{2r}(x) := \sum_{k=0}^r \frac{(-1)^k}{4^k k!} H_{2k}(s) = \sum_{j=0}^r (-1)^j b_j \frac{x^{2j}}{2j!}$$

From the expansion of Hermite polynomials (see e.g. [], )

$$H_{2k}(s) =$$

we deduce

$$b_j := \sum_{k=j}^r \frac{2k!}{4^{k-j} k! (k-j)!} = \frac{j!}{4^{r-j}} \sum_{k=j}^r 4^{r-k} \binom{2k}{k} \binom{k}{j}.$$

Setting  $j := r - p$ , we can write as follows the expression of the theorem

$$\tilde{H}_{2r}(x) = \frac{(2r+1)!}{r!} \sum_{j=0}^r \frac{(-1)^j}{4^{r-j}(r-j)!} \frac{x^{2j}}{(2j+1)!} = \sum_{j=0}^r (-1)^j a_j \frac{x^{2j}}{2j!}$$

where

$$a_j := \frac{(2r+1)!}{r!} \frac{1}{4^{r-j}(r-j)!} \frac{1}{2j+1}$$

Therefore, we have to prove that  $a_j = b_j$  for  $0 \leq j \leq r$ , i.e., after simplification:

$$\alpha_j := \frac{(2r+1)!}{r!} \frac{1}{(r-j)!} \frac{1}{2j+1} = \beta_j := j! \sum_{k=j}^r 4^{r-k} \binom{2k}{k} \binom{k}{j}$$

This can be proved by induction on  $r$ . For  $r = 0$ , we have  $j = 0$ , thus  $\alpha_0 = \beta_0 = 1$ . For  $r = 1$ , we get

$$\begin{aligned} \alpha_0 &= 3! = \beta_0 = \sum_{k=0}^1 4^{1-k} \binom{2k}{k} = 4 + 2, \\ \alpha_1 &= 3! \frac{1}{3} = 2 = \beta_1 = \binom{2}{1} = 2 \end{aligned}$$

Assume that the property is true for the coefficients  $0 \leq j \leq r$  of  $\tilde{H}_{2r}$  and let us prove it for  $\tilde{H}_{2r+2}$ , i.e. that  $\alpha_j^{[r+1]} = \beta_j^{[r+1]}$  holds for  $0 \leq j \leq r+1$ . We have respectively

$$\begin{aligned} \alpha_j^{[r+1]} &:= \frac{(2r+3)!}{(r+1)!} \frac{1}{(r+1-j)!} \frac{1}{2j+1} \\ \beta_j^{[r+1]} &= j! \sum_{k=j}^{r+1} 4^{r+1-k} \binom{2k}{k} \binom{k}{j} = 4\beta_j^{[r]} + j! \binom{2r+2}{r+1} \binom{r+1}{j} \\ &= \frac{(2r+1)!}{r!} \frac{1}{(r-j)!} \frac{4}{2j+1} + \frac{(2r+2)!}{(r+1)!(r+1-j)!} \\ &= \frac{(2r+1)!}{(r+1)!(r+1-j)!} \frac{1}{2j+1} (4(r+1)(r+1-j) + 2(r+1)(2j+1)) \\ &= \frac{(2r+1)!}{(r+1)!(r+1-j)!} \frac{(2r+2)(2r+3)}{2j+1} \\ &= \frac{1}{2j+1} \frac{(2r+3)!}{(r+1)!(r+1-j)!} = \alpha_j^{[r+1]}, \quad \text{q.e.d. } \square \end{aligned}$$

**Remark.** It would be possible to express Weierstrass left quasi-interpolants in delta-form

$$W_n^{[r]} = V_n^{[r]} W_n = W_n + \sum_{\ell=1}^r \beta_\ell(n) \frac{\delta^{2\ell}}{2\ell!} W_n$$

However, this will not be developed here.



## 5 Right $W_n$ -quasi-interpolants on polynomials

**Definition.** Considering the partial sums of order  $r$  of the inverse  $W$ -operator

$$V_n^{[r]} := \sum_{k=0}^r \frac{1}{n^k} \frac{(-1)^k}{4^k k!} D^{2k}$$

one defines the right Weierstrass quasi-interpolants of order  $r$  as follows:

$$\overline{W}_n^{[r]} f := W_n V_n^{[r]} f := W_n \left( \sum_{k=0}^r \frac{1}{n^k} \frac{(-1)^k}{4^k k!} D^{2k} f \right)$$

### 5.1 Representation of $\overline{W}_n^{[r]}$ in $D$ -form

Using the first form of  $W_n f$ , we get

$$\overline{W}_n^{[r]} f(x) = \sqrt{\frac{n}{\pi}} \int_{-\infty}^{+\infty} \exp(-n(x-t)^2) V_n^{[r]} f(t) dt$$

Using the second form, we get

$$W_n f := \sqrt{\frac{n}{\pi}} \int_{-\infty}^{+\infty} f(x-t) \exp(-nt^2) dt = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} f(x-s/\sqrt{n}) \exp(-s^2) ds$$

we get the representations

$$\overline{W}_n^{[r]} f(x) = \sqrt{\frac{n}{\pi}} \int_{-\infty}^{+\infty} \exp(-nt^2) V_n^{[r]} f(x-t) dt = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \exp(-s^2) V_n^{[r]} f(x-s/\sqrt{n}) ds$$

### 5.2 Representation of $\overline{W}_n^{[r]}$ in $\delta$ -form

$$V_n = \sum_{\ell \geq 0} \alpha(n, \ell) \frac{h^\ell \delta^\ell}{\ell!}$$

Using the first form of  $W_n f$ , we get

$$\overline{W}_n^{[r]} f(x) = \sqrt{\frac{n}{\pi}} \int_{-\infty}^{+\infty} \exp(-n(x-t)^2) V_n^{[r]} f(t) dt$$

Using the second form of  $W_n f$ , we get the representations

$$\overline{W}_n^{[r]} f(x) = \sqrt{\frac{n}{\pi}} \int_{-\infty}^{+\infty} \exp(-nt^2) V_n^{[r]} f(x-t) dt = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \exp(-s^2) V_n^{[r]} f(x-s/\sqrt{n}) ds$$

**Remark.** The right  $W_n$ -quasi-interpolants in  $D$ -form seem to be less interesting than the left ones since they explicitly use the derivatives of the function to be approximated. Yet the representation in  $\delta$ -form seem to be more interesting since it only uses values of  $f$  at integer points. However, we do not develop their study in the present paper and we postpone it to a future work.

## 6 Norms of the Weierstrass left Quasi-Interpolants

From the expression

$$W_n^{[r]}f(x) := \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \exp(-s^2) \tilde{H}_{2r}(s) f(x - s/\sqrt{n}) ds$$

we first deduce the majoration

$$|W_n^{[r]}f(x)| \leq \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \exp(-s^2) |\tilde{H}_{2r}(s)| ds, \quad \|f\|_{\infty} = 1$$

### 6.1 Exact value of the norm

**Theorem 5.** *The infinite norm of  $W_n^{[r]}$  is given by the expression*

$$N_r := \|W_n^{[r]}\|_{\infty} = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \exp(-s^2) |\tilde{H}_{2r}(s)| ds$$

*Proof.* Let  $f$  be the function defined by  $f(x) = \text{sgn}(\tilde{H}_{2r}(s))$ , then  $\|f\|_{\infty} = 1$  and  $W_n^{[r]}f(x) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \exp(-s^2) |\tilde{H}_{2r}(s)| ds = N_r$ , q.e.d.  $\square$

### 6.2 Strong majoration of the norm

From [10] (chapter 1, p.31), we know that

$$e^{-t^2/2} |H_{2p}(t)| \leq 2^p \sqrt{2p!}$$

and we deduce

$$e^{-t^2/2} |\tilde{H}_{2r}(t)| \leq \sum_{p=0}^r \frac{1}{4^p p!} e^{-t^2/2} |H_{2p}(t)| \leq \sigma_r := \sum_{p=0}^r \frac{\sqrt{2p!}}{2^p p!}$$

which leads to the following majoration of the norm

**Theorem 6.** *The norm  $\|W_n^{[r]}\|_{\infty}$  is majorised by the following constant independent of  $n$ :*

$$\|W_n^{[r]}\|_{\infty} \leq C_r := r + \sqrt{2}$$

*Proof.* Using the above majoration, we obtain

$$N_r = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \exp(-s^2) |\tilde{H}_{2r}(s)| ds \leq \frac{\sigma_r}{\sqrt{\pi}} \int_{\mathbb{R}} \exp(-s^2/2) ds = \sqrt{2} \sigma_r$$

The quantity  $\sigma_r$  can be majorised as follows

$$\sigma_r = 1 + \sum_{p=1}^r \sqrt{\left( \frac{1.3 \dots 2p-1}{2.4 \dots 2p} \right)} \leq 1 + \frac{r}{\sqrt{2}}$$

Therefore we finally obtain

$$\sqrt{2}\sigma_r \leq C_r = r + \sqrt{2} \quad \text{q.e.d.} \quad \square$$

**Table of the first values of  $N_r$  and  $C_r$**

$r$	1	2	3	4	5	6	7	8	9	10
$N_r$	1.14	1.22	1.28	1.33	1.37	1.40	1.43	1.45	1.47	1.49
$\sqrt{2}\sigma_r$	2.41	3.28	4.07	4.81	5.51	6.18	6.83	7.46	8.07	8.66
$C_r$	2.41	3.41	4.41	5.41	6.41	7.41	8.41	9.41	10.41	11.41

We can observe the slow increase of the norms of Weierstrass quasi-interpoants.

## 7 Convergence properties of WQIs

From the definition

$$W_n f(x) := \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} f(x + s/\sqrt{n}) \exp(-s^2) ds$$

or the properties of the heat kernel, it is clear that

$$\lim_{n \rightarrow +\infty} W_n f(x) = f(x) \quad \forall f \in C(\mathbb{R})$$

The same result holds for the WQIs since  $W_n^{[r]} f(x)$  is a finite linear combination of derivatives of  $W_n f$  with coefficients tending to zero when  $n \rightarrow \infty$ . The problem is to show that the convergence order is improved for smooth functions.

### 7.1 Convergence order of the operator $W_n$

For  $f \in C^3(\mathbb{R})$ , we use the Taylor expansion

$$f(x + s/\sqrt{n}) = f(x) + \frac{s}{\sqrt{n}} f'(x) + \frac{s^2}{2n} f''(x) + r_n(x, s)$$

where  $r_n(x, s) := \frac{1}{2} \int_x^{x+s/\sqrt{n}} (x + s/\sqrt{n} - u)^2 f^{(3)}(u) du$ , and as  $M_2 = \frac{1}{2}$ , we deduce

$$W_n f(x) = f(x) + \frac{1}{4n} f''(x) + R_n(x), \quad R_n(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} r_n(x, s) \exp(-s^2) ds$$

Now, we majorize the expression

$$n(W_n f(x) - f(x)) - \frac{1}{4} f''(x) = nR_n(x)$$

Using the change of variable  $u = x + t/\sqrt{n}$ ,  $t \in [0, s]$ , we get

$$nR_n(x) = \frac{1}{2\sqrt{\pi n}} \int_{-\infty}^{+\infty} \left( \int_0^s (s-t)^2 f^{(3)}(x+t/\sqrt{n}) dt \right)$$

and also the majoration

$$\left| \int_0^s (s-t)^2 f^{(3)}(x+t/\sqrt{n}) dt \right| \leq \|f^{(3)}\|_{\infty} \int_0^{|s|} (|s|-t)^2 ds = \|f^{(3)}\|_{\infty} \frac{1}{3} |s|^3$$

By Lemma 1, we know that  $M_3 = \frac{1}{\sqrt{\pi}}$ , thus we get

$$|nR_n(x)| \leq \frac{1}{6\pi\sqrt{n}} \|f^{(3)}\|_{\infty} \Rightarrow \lim nR_n(x) = 0,$$

and we finally obtain

$$\lim_{n \rightarrow \infty} n(W_n f(x) - f(x)) = \frac{1}{4} f''(x)$$

In a similar way, one obtains successively

$$\lim n^2(W_n^{[1]} f(x) - f(x)) = -\frac{1}{32} f^{(4)}(x)$$

and

$$\lim n^3(W_n^{[2]} f(x) - f(x)) = \frac{1}{384} f^{(6)}(x)$$

for  $f \in W^{5,\infty}(\mathbb{R})$  and  $f \in W^{7,\infty}(\mathbb{R})$  respectively. Let us now study the general case.

## 7.2 The remainder $g - W^{[r]}g$ on polynomials

The aim of this section is to give an expression of the remainder  $g - W^{[r]}g$  for  $g \in \mathbb{P}$ .

$$V_n = \sum_{k \geq 0} a_k(n) D^{2k}, \quad V_n^{[r]} = \sum_{k=0}^r a_k(n) D^{2k}, \quad a_k(n) = (-1)^k / (4n)^k k!$$

Therefore, as  $V_n W_n = Id$ ,

$$g - W^{[r]}g = (V_n - V_n^{[r]})W_n g = \sum_{k \geq r+1} a_k(n) D^{2k} W_n g$$

On the other hand, we have

$$D^{2k} W_n g = \sum_{\ell \geq 0} b_{\ell}(n) D^{2(k+\ell)} g, \quad b_{\ell}(n) = 1/(4n)^{\ell} \ell!$$

thus we get

$$g - W^{[r]}g = \sum_{\ell \geq 0} \sum_{k \geq r+1} a_k(n) b_\ell(n) D^{2(k+\ell)} g = \sum_{m \geq r+1} \left( \sum_{k+\ell=m, k \geq r+1} a_k(n) b_\ell(n) \right) D^{2m} g$$

Let us denote by  $c_m(n)$  the coefficient of  $D^{2m}g$ . For example, we get

$$c_{r+2}(n) = a_{r+2}(n) + a_{r+1}(n)b_1(n) = \frac{(-1)^{r+1}}{(4n)^{r+2}(r+2)!} \binom{r+1}{1}$$

$$c_{r+3}(n) = a_{r+3}(n) + a_{r+2}(n)b_1(n) + a_{r+1}(n)b_2(n) = \frac{(-1)^{r+1}}{(4n)^{r+3}(r+3)!} \binom{r+2}{2}$$

More generally, for all  $p \in \mathbb{N}$ , using the identity (see [18], Section 4.2.1, formula 4)

$$\sum_{i=0}^{p-1} (-1)^i \binom{r+p}{i} = (-1)^{p-1} \binom{r+p-1}{p-1}$$

we obtain

$$c_{r+p}(n) = \sum_{k=r+1}^{r+p} a_k(n) b_{r+p-k}(n) = \frac{(-1)^{r+p}}{(4n)^{r+p}(r+p)!} \sum_{i=0}^{p-1} (-1)^i \binom{r+p}{i}$$

hence

$$c_{r+p}(n) = \frac{(-1)^{r+1}}{(4n)^{r+p}(r+p)!} \binom{r+p-1}{p-1} = (-1)^{p+1} \binom{r+p-1}{p-1} a_{r+p}(n)$$

Therefore we obtain

**Theorem 7.** *For any polynomial  $g$ , there holds the following representation of the error for the Weierstrass quasi-interpolant  $W^{[r]}$ :*

$$g - W^{[r]}g = \sum_{p \geq 1} (-1)^{p-1} \binom{r+p-1}{p-1} a_{r+p}(n) D^{2(r+p)} g$$

**Remarks.**

- 1) This is another proof of the fact that  $W^{[r]}$  is exact on  $\mathbb{P}_{2r+1}$ .
- 2) Moreover, we deduce that

$$n^{r+1}(g - W^{[r]}g) = n^{r+1}a_{r+1}(n)D^{2(r+1)}g + \sum_{p \geq 2} (-1)^{p-1} \binom{r+p-1}{p-1} n^{r+1}a_{r+p}(n)D^{2(r+p)}g$$

where the sum is finite since  $g \in \Pi$ . We also have

$$n^{r+1}a_{r+1}(n) = \frac{(-1)^{r+1}}{4^{r+1}(r+1)!} \quad \text{and, for } p \geq 2, \quad n^{r+1}a_{r+p}(n) = \frac{(-1)^{r+p}}{4^{r+p}(r+p)! n^{p-1}}$$

As  $\lim_{n \rightarrow \infty} n^{r+1}a_{r+p}(n) = 0$  for  $p \geq 2$ , we obtain, for any polynomial  $g$ :

$$\lim_{n \rightarrow \infty} n^{r+1}(g - W^{[r]}g) = \frac{(-1)^{r+1}}{4^{r+1}(r+1)!} D^{2r+2}g$$

### 7.3 Convergence order of the quasi-interpolant $W_n^{[r]}$

In the general case, we have the following Voronovskaja-type result

**Theorem 8.** *Assume that  $f$  has a bounded derivative of order  $2r + 3$ . Then the following limit holds*

$$\lim_{n \rightarrow \infty} n^{r+1}(f(x) - W^{[r]}f(x)) = \frac{(-1)^{r+1}}{4^{r+1}(r+1)!} D^{2r+2}f(x)$$

*Proof.* We start from the expression

$$W_n^{[r]}f(x) := \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \exp(-s^2) \tilde{H}_{2r}(s) f(x + s/\sqrt{n}) ds$$

Taylor's formula gives

$$f(x + s/\sqrt{n}) = f(x) + \sum_{k=1}^{2r+2} \frac{s^k}{n^{k/2}} \frac{f^{(k)}(x)}{k!} + g_n(x, s)$$

where

$$g_n(x, s) = \frac{1}{(2r+2)!} \int_x^{x+s/\sqrt{n}} (x + s/\sqrt{n} - u)^{2r+2} D^{2r+3}f(u) du$$

For all  $1 \leq k \leq 2r + 2$ , we compute the values of

$$M_k^{[r]} := \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \exp(-s^2) s^k \tilde{H}_{2r}(s) ds$$

in function of the moments of Hermite polynomials:

$$\mu_{2r,k} = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \exp(-s^2) s^k H_{2r}(s) ds$$

which are equal to zero for  $0 \leq k \leq 2r - 1$ , and (see [1], formula 22.13.19, p.786, for  $x = 1, P(1) = 1$ ):

$$\mu_{2r,2r} := \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \exp(-s^2) s^{2r} H_{2r}(s) ds = r!$$

Therefore, by definition of  $\tilde{H}_{2r}$ , one has for all  $k \geq$ :

$$M_k^{[r]} := \frac{1}{\sqrt{\pi}} \sum_{j=0}^r \frac{(-1)^j}{4^j j!} \int_{\mathbb{R}} \exp(-s^2) s^k H_{2j}(s) ds$$

Since  $H_{2j}(x)$  is a polynomial of even degree, it is clear that  $M_k^{[r]} := 0$  for all  $k$  odd. For  $k = 2p$ , one has

$$M_{2p}^{[r]} := \sum_{j=0}^r \frac{(-1)^j}{4^j j!} \mu_{2j,2p}$$

Using Maple or [1] as above, we get

$$\mu_{2j,2p} := \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \exp(-s^2) s^{2p} H_{2j}(s) ds = 4^j j! \binom{p}{j} \Gamma(p+1/2) / \sqrt{\pi}$$

from which we deduce, for  $p \leq r$ :

$$M_{2p}^{[r]} = \sum_{j=0}^r \frac{(-1)^j}{4^j j!} \mu(2j, 2p) = \frac{1}{\sqrt{\pi}} \Gamma(p+1/2) \sum_{j=0}^r (-1)^j \binom{p}{j} = (-1)^r \binom{p-1}{r} \frac{1}{\sqrt{\pi}} \Gamma(p+1/2) = 0$$

and for  $p = r+1$ :

$$M_{2r+2}^{[r]} = (-1)^r \frac{1}{\sqrt{\pi}} \Gamma(r+3/2) = (-1)^r \frac{1 \cdot 3 \cdot 5 \dots (2r+1)}{2^{r+1}} = (-1)^r \frac{(2r+2)!}{4^{r+1}(r+1)!}$$

From that we deduce, after simplification,

$$W^{[r]} f(x) = f(x) + (-1)^r \frac{D^{(2r+2)} f(x)}{4^{r+1}(r+1)!} \mu_{2r,k} + R_n(x)$$

with

$$R_n(x) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \exp(-s^2) \tilde{H}_{2r}(s) g_n(x, s) ds$$

In order to get the result, we have still to prove that

$$\lim_{n \rightarrow +\infty} n^{r+1} R_n(x) = 0$$

A first majoration gives

$$n^{r+1} |g_n(x, s)| \leq \frac{1}{\sqrt{n}} \frac{1}{(2r+2)!} \|D^{2r+3} f\|_{\infty} \int_0^{|s|} (|s| - t)^{2r+2} dt$$

Therefore, as the integral is equal to  $|s|^{2r+3}/(2r+3)$ , we get the second majoration

$$n^{r+1} |R_n(x)| \leq \frac{1}{\sqrt{n}} \frac{1}{(2r+3)!} \|D^{2r+3} f\|_{\infty} \int_{\mathbb{R}} \exp(-s^2) |\tilde{H}_{2r}(s)| |s|^{2r+3} ds$$

It remains to prove that the integral is bounded independently of  $n$ . This is true because first we know (Section 6.2) that  $e^{-s^2} |\tilde{H}_{2r}(s)| \leq \sigma_r e^{-s^2/2}$ , thus

$$\int_{\mathbb{R}} \exp(-s^2) |\tilde{H}_{2r}(s)| |s|^{2r+3} ds \leq \sigma_r \int_{\mathbb{R}} \exp(-s^2/2) |s|^{2r+3} ds$$

which is proportional to an odd moment of the function  $e^{-s^2}$  (Lemma 1), the coefficient depending on  $r$ , but not on  $n$ . Finally, we get

$$n^{r+1} |R_n(x)| \leq \frac{C_r}{\sqrt{n}} \Rightarrow \lim_{n \rightarrow +\infty} n^{r+1} |R_n(x)| = 0, \text{ q.e.d.}$$

## 8 A short note on the computation of W-quasi-interpolants

1) The first form of  $W_n^{[r]}$  leads to computing the following integral for any  $x \in \mathbb{R}$

$$W_n^{[r]}f(x) = \sqrt{\frac{n}{\pi}} \int_{-\infty}^{+\infty} \tilde{H}_{2k}(x-t) \exp(-n(x-t)^2) f(t) dt$$

2) On the other hand, the second form leads to the computation of the integral

$$W_n^{[r]}f(x) := \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \exp(-s^2) \tilde{H}_r(s) f(x - s/\sqrt{n}) ds$$

Using a quadrature formula of type

$$\int_{\mathbb{R}} e^{-x^2} f(x) dx \approx \sum_{i=1}^k A_i f(x_i)$$

this second form is substituted for the approximate discrete operator

$$\overline{W}_n^{[r]}f(x) := \frac{1}{\sqrt{\pi}} \sum_{i=1}^k A_i \tilde{H}_r(x_i) f(x - x_i/\sqrt{n})$$

For given values of  $(r, k)$ , the values of  $\tilde{H}_r(x_i)$  can be computed once for all in advance. Then, one can vary the value of  $n$ .

The quadrature formulas can be either the Gauss-Hermite ones  $([1, 10])$  or the trapezoidal formula in  $\mathbb{R}$ . In Henrici [11], chapter 11, one may find good reasons for using the quadrature formula

$$Qf(h) = h \sum_{k \in \mathbb{Z}} \exp(-k^2 h^2) f(kh) \approx \int_{\mathbb{R}} \exp(-t^2) f(t) dt$$

This topic will be developed in a further paper together with numerical examples and some applications.

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